Intro GGT — Homework 1

Question 1. Let G be a group and $H \leq G$. Define an action of G on G/H by

$$g \cdot (aH) = (ga)H.$$

1. Prove that the above formula gives a well-defined group action.
2. Show that the action is transitive.
3. Compute the stabilizer of the coset $H \in G/H$; of any coset $aH \in G/H$.
4. Show that the action is faithful if and only if
$$\bigcap_{g \in G} gHg^{-1} = \{e\}.$$
5. Show that if H is a normal subgroup, then the action $G \curvearrowright G/H$ induces naturally an action $G/H \curvearrowright G/H$. What is this action?
6. Let $G \overset{\alpha}{\curvearrowright} X$ and $G \overset{\beta}{\curvearrowright} Y$ be two actions. A function $\varphi\colon X \to Y$ is called *G-equivariant* if $\varphi(gx) = g\varphi(x)$ for all $g \in G$ and $x \in X$. Show that $\mathrm{Stab}_\alpha(x) \subset \mathrm{Stab}_\beta(\varphi(x))$ for all $x \in X$.
7. Show that if $\varphi\colon X \to Y$ is G-equivariant and bijection, then φ^{-1} is also G-equivariant. In this case, φ is called a *G-action isomorphism*, and the two actions are said to be *isomorphic*.
8. Let $G \curvearrowright X$ be a transitive action and $x \in X$. Prove that the map
$$\begin{aligned} \varphi\colon G/\mathrm{Stab}(x) &\to X \\ g\mathrm{Stab}(x) &\mapsto g \cdot x \end{aligned}$$
is a G-action isomorphism.
9. Conclude that every transitive action of G is isomorphic to an action of the form $G \curvearrowright G/H$.
10. Show that two actions $G \curvearrowright G/H$ and $G \curvearrowright G/K$ are isomorphic if and only if H and K are conjugate in G.